

Regular isolated black holes

Carlos Kozameh¹, Osvaldo Moreschi¹ and Alejandro Perez²

²*Centre de Physique Théorique, Campus de Luminy, 13288 Marseille, France. and*

¹*FaMAF, Instituto de Física Enrique Gaviola (IFEG), CONICET, Ciudad Universitaria, (5000) Córdoba, Argentina.*

(Dated: March 21, 2011)

We define a family of spacetimes representing isolated black holes exhibiting remarkable universal properties which are natural generalizations from stationary spacetime. They admit a well defined notion of surface gravity k_H . This generalized surface gravity mediates an exponential relation between a regular null coordinate w near the horizon and an asymptotic Bondi null coordinate u defined in the vicinity of future null infinity. Our construction provides a framework for the study of gravitational collapse of an isolated system in its late stage of evolution.

Black holes are intriguing solutions of classical general relativity describing important aspects of the physics of gravitational collapse. Their existence in our nearby universe is by now supported by a great amount of observational evidence [1–3]. When isolated, these systems are expected to be simple for late and distant observers. The general belief is that, once the initial very dynamical phase of collapse has passed, the system should settle down to a stationary situation completely described by the Kerr solution.

However, general relativity is a global theory. A spacetime that settles down to a stationary situation is different from a stationary one. The typical collapse is expected to be more complicated with incoming and outgoing radiation, and possible matter accretion. This will produce a space time that only asymptotically resembles a stationary black hole. In this work we introduce a framework to study black holes in their late stage of dynamical evolution once all the matter has fallen into the black hole but still taking into account incoming and outgoing radiation.

Our analysis is based on the assumption that the null surfaces associated to suitably chosen (inertial) retarded time u at future null infinity define a smooth null foliation in the vicinity of the black hole event horizon. More precisely, there is a null physical coordinate $w(u)$ (unique up to scaling) that can be used to describe fields in the vicinity of the horizon. The above assumption, made precise in what follows, characterizes the spacetimes studied in this letter which are referred to as *isolated black holes* (IBH).

For IBHs the following remarkable properties hold.

1. There is an exponential relationship between the null function w and a preferred retarded time u function.
2. There exists a geometrically defined smooth vector field that is null both at the horizon H and at future null infinity.
3. There is a generalized notion of surface gravity which is constant on H .

There has been other approaches to this problem both at a global and local level. In 1967 Pajerski and

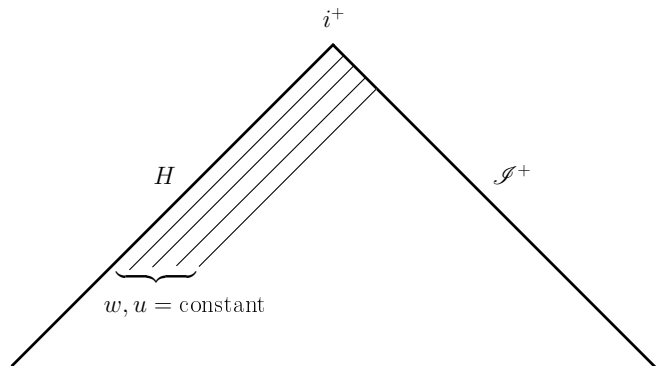


FIG. 1. An isolated black hole spacetime. In a vicinity of i^+ , the null surfaces of constant retarded time u are smooth all the way up to the event horizon for IBHs.

Newman[4] proposed a formulation based on the spin coefficient formalism to generalize the Schwarzschild black hole solution. However, an assumption on the vanishing of a spin coefficient turned out to be too restrictive. More recently, Ashtekar and collaborators[5] gave a local definition of the so-called *isolated horizons* which, as in the stationary case, assumes the vanishing of the shear and divergence on the null surface. This definition is also restrictive in the sense that it implies no incoming radiation to the surface. The framework presented here aims at overcoming these difficulties.

The setup

Consider (\mathcal{M}, g_{ab}) an asymptotically flat spacetime at future null infinity containing a black hole. Its conformal diagram is depicted in Figure 1. In the past of an open set of future null infinity (\mathcal{J}^+)—defined by those points for which their Bondi¹ retarded time u is in the range $u \in (u_0, \infty)$ —we require the existence of a regular

¹ A Bondi retarded time u is such that the sections $u=\text{constant}$ at \mathcal{J}^+ (referred to as *Bondi cuts*) have an intrinsic metric given by (minus) the metric of the unit sphere.

null function $w(u)$ such that: $w = 0$ at the horizon H , and $w < 0$ in the region of interest. Clearly there is a large freedom in selecting the function $w(u)$ if the only requirement is to satisfy the above condition. In what follows we completely reduce this freedom by selecting a unique (up to constant scaling) null function $w(u)$ using the available structure at \mathcal{I}^+ . We can now list the ingredients of our construction.

1. (\mathcal{M}, g_{ab}) is an asymptotically flat spacetime at future null infinity containing a black hole, assumed to be vacuum in the vicinity of i^+ . The topology of the BH event horizon H is assumed to be $S^2 \times \mathbb{R}$ in that region.
2. We assume the existence of a null function u in the asymptotic region such that the cuts at \mathcal{I}^+ are Bondi. We introduce the one form $\tilde{\ell}_a \equiv (du)_a$, then the vector field $\tilde{\ell}^a$ defines a null geodesic congruence. It is therefore natural to introduce the affine function r through $\tilde{\ell} = \frac{\partial}{\partial r}$.
3. In a space-time satisfying the previous conditions there is a three parameter family of Bondi systems such that the leading order behaviour of the shear of the congruence defined by $\tilde{\ell}$ vanishes in the limit $u \rightarrow \infty$. We completely eliminate this freedom by requiring that the cuts $u = \text{constant}$ asymptotically coincide with the *center of mass*² in the same regime.
4. The functions (u, r) can be used as coordinate functions in the region in which the null congruence $\tilde{\ell}$ does not show caustics. The asymptotically flat region is reached in the limit $r \rightarrow \infty$. The origin of this affine parameter is chosen so that r asymptotically coincides with the luminosity distance (i.e. the expansion of the null congruence $\tilde{\ell}$ goes as $r^{-1} + \mathcal{O}(r^{-3})$).
5. There exists a smooth null function $w = w(u)$ such that
 - (a) $w = 0$ at the horizon H .
 - (b) $\dot{w} \equiv \frac{dw}{du} > 0$.
 - (c) $w < 0$ for all u .
 - (d) $\lim_{u \rightarrow \infty} w = 0$.

² Notice that in an asymptotically flat spacetime the Bondi flux of gravitational radiation must go to zero as $u \rightarrow \infty$. Although the notion of angular momentum and center of mass in general relativity are very complex [6, 7], the situation drastically simplifies when the radiation vanishes. In fact, total angular momentum behaves just as in special relativity under translations $J^{ab} \rightarrow J^{ab} + P^{[a} X^{b]}$; therefore, one can eliminate the translational freedom by placing the origin of the reference frame on the center of mass world line. Analogously, this corresponds to the selection of the *center of mass* sections at \mathcal{I}^+ . [6]

6. We define the one form $\ell_a \equiv (dw)_a$, then, the geodesic vector field ℓ^a is also tangent to the null congruence defined in point 2. It is therefore natural to introduce the affine function y through $\ell = \frac{\partial}{\partial y}$.
7. The functions (w, y) can be used as coordinate functions in the region where the null congruence ℓ does not show caustics.
8. Given the functions (u, r) one can choose the affine parameter y , for each null geodesic, so that the 2-surfaces $u = \text{const.}$, $r = \text{const.}$ coincide with the 2-surfaces $w = \text{const.}$, $y = \text{const.}$ This implies the following relationship between r and y :

$$r = \dot{w}y + r_0(w). \quad (1)$$

One then completes a coordinate system by choosing angular sphere-coordinates (θ, ϕ) or stereographic sphere-coordinates $(\zeta, \bar{\zeta})$.

9. In the exterior of the BH there is clearly a smooth relationship between the pairs (u, r) and (w, y) . We assume that r is a smooth function of (w, y) all the way up to the horizon.

Let us observe that from the null vector fields ℓ^a and $\tilde{\ell}^a$ one can construct null tetrads $(\ell^a, m^a, \bar{m}^a, n^a)$, and $(\tilde{\ell}^a, \tilde{m}^a, \tilde{\bar{m}}^a, \tilde{n}^a)$ adapted to the geometry of the coordinate system introduced above.³ The freedom in this choice is reduced by choosing the vectors $m^a = \tilde{m}^a$ and tangent to the topological 2-spheres $(w, y) = \text{constant} = (u, r)$.

From $w = w(u)$ it follows that $dw = \dot{w} du$ which implies the following relation between the two tetrads

$$\ell^a = \dot{w} \tilde{\ell}^a, \quad n^a = \frac{1}{\dot{w}} \tilde{n}^a, \quad m^a = \tilde{m}^a. \quad (2)$$

If we denote the five (complex) Weyl tensor null tetrad components [8] Ψ_N and $\tilde{\Psi}_N$ for $N \in \{0, 1, 2, 3, 4\}$ in each of the respective tetrads, then we get the following relations

$$\Psi_N = \dot{w}^{(2-N)} \tilde{\Psi}_N. \quad (3)$$

Before proceeding further we would like to give a motivation for assumption 9. From the previous relation one has

$$\Psi_2(w, y, x^A) = \tilde{\Psi}_2(u(w), r(w, y), x^A); \quad (4)$$

where (x^A) denotes angular coordinates. Therefore, since the left hand side is a regular expression in terms of the coordinate w , it is natural for r to be a regular function of w as it was assumed in 9.

³ With the usual normalization $1 = \ell^a n_a = -m^a \bar{m}_a$ and $1 = \tilde{\ell}^a \tilde{n}_a = -\tilde{m}^a \tilde{\bar{m}}_a$ with all other respective scalar products being zero.

Consequences

The first consequence of the regularity property is that the limit $r_H \equiv \lim_{w \rightarrow 0} r(w, y)$ exists. Another consequence is that $\dot{w}(w)$ and $r_0(w)$, as defined in equation (1), are regular functions of w . Moreover, we have

$$w(u) = \int_{-\infty}^u \dot{w}(u') du' \Rightarrow \dot{w}(0) = \lim_{u \rightarrow \infty} \dot{w}(u) = 0. \quad (5)$$

It then follows from (1) that

$$r_H = \text{constant}. \quad (6)$$

This allows to define r_H as the radius of the isolated black hole. Since by assumption $\dot{w}(w)$ admits a Taylor expansion around $w = 0$ we can write.

$$\dot{w} = aw + \mathcal{O}(w^2). \quad (7)$$

Assuming that $a \neq 0$ the above equation can be integrated giving the important relation

$$\boxed{w(u) = -\exp(a(u - u_0)) + \mathcal{O}(\exp(2au))}, \quad (8)$$

where $\exp(-au_0)$ is the rescaling freedom mentioned previously associated with the choice of origin for the Bondi retarded time u . Note also that in order to satisfy (5) one has $a < 0$.

A natural question arises: is it possible to get more information concerning the nature of the coefficient a ? Next we show that a has a clear geometrical meaning. We start from the vector field

$$\chi \equiv \frac{\partial}{\partial u}. \quad (9)$$

The vector field χ has several useful properties:

1. It is a smooth vector field that is a null geodesic generator at \mathcal{I}^+ . As u is a Bondi coordinate it generates inertial time translations at future null infinity.
2. It is a null geodesic generator of the horizon H .
3. At the horizon H , χ satisfies the equation,

$$\chi^a \nabla_a \chi^b \equiv k_H \chi^b;$$

where k_H is a generalized surface gravity.

4.

$$\boxed{k_H = \text{const.} = -a}. \quad (10)$$

The previous statements follow from expressing χ in terms of the regular coordinates $(w, y, \zeta, \bar{\zeta})$, namely

$$\begin{aligned} \chi &= \dot{w} \frac{\partial}{\partial w} - \frac{\partial r}{\partial w} \frac{\partial}{\partial y} \\ &= aw \frac{\partial}{\partial w} - \left(ay + \frac{dr_0}{dw} \right) \frac{\partial}{\partial y} + \mathcal{O}(w^2). \end{aligned} \quad (11)$$

Evaluating the previous equation at $w = 0$ one obtains

$$\chi|_{w=0} = - \left(ay + \frac{dr_0}{dw} \Big|_{w=0} \right) \frac{\partial}{\partial y}, \quad (12)$$

Eq. (12) implies properties 2 to 4.

In this way, the class of spacetimes considered here admits a notion of surface gravity which coincides with the usual one in cases when the spacetime is stationary, e.g. a member of the Kerr family. Note that if we had taken $a = 0$ above, one would have obtained $k_H = 0$. This case corresponds to the especial cases involving (in particular) the stationary extremal black holes.

With this definition of surface gravity, the relation between the null coordinate w and the Bondi retarded time u reads

$$\boxed{w = -\exp(-k_H(u - u_0)) + \mathcal{O}(\exp(-2k_H u))}; \quad (13)$$

which we recognize as the generalization of the smooth Kruskal coordinate transformation that can be found in Schwarzschild and Kerr geometries.

Finally, we want to address the issue of characteristic data in our framework; which is $\Psi_0(y, \theta, \phi)$ at H and $\Psi_4^0(w, \theta, \phi)$ at \mathcal{I}^+ . At the horizon H one has to choose Ψ_0 going to zero as $y \rightarrow \infty$. A necessary condition for this behaviour is that Ψ_0 goes to zero at least as $O(1/y^3)$. This comes from the requirement that the area of the sections $y=\text{constant}$ must go to a finite value when $y \rightarrow \infty$. At future null infinity a physical condition is that the radiation field $\tilde{\Psi}_4^0$ must go to zero as $u \rightarrow \infty$.

However, our characteristic data Ψ_4^0 could be unbounded as $w \rightarrow 0$; since it can be shown that it is related to the radiation field via $\Psi_4^0 = \tilde{\Psi}_4^0/\dot{w}^3$. This is the case for radiation fields predicted in studies of gravitational collapse in perturbation theory where $\tilde{\Psi}_4^0$ has a power-law fall-off behaviour as $u \rightarrow \infty$ [9]. It might appear that such unbounded data could invalidate some of the assumptions in the definition of an IBH. Nevertheless, just from the physical condition that $\tilde{\Psi}_4^0$ vanishes as $u \rightarrow \infty$, one can prove that the null geodesic congruence used in our construction is caustic free in the region of interest. Therefore, the power law fall-off behaviour for the characteristic data expected in gravitational collapse is admitted in our framework.

Summary and final comments

In this work we have introduced a framework to study black holes in their late stage of dynamical evolution, that is, after all matter has fallen into the BH but still taking into account both incoming and outgoing radiation. Our framework is sufficiently general to include physically interesting collapsing scenarios. The construction of two related geometrical null coordinate systems adapted to the available structure, plus mild regularity conditions at the horizon lead to novel results: physical coordinates (and associated null tetrads) to study the field equations

in the late phase of collapse, a generalized surface gravity which is constant (zeroth law of black hole mechanics), the correct exponential behaviour between w and u (mediated by the surface gravity) which should be relevant for the study of Hawking BH radiation.

In future work we will explicitly analyze the Einstein field equations near the horizon H , expecting to obtain more dynamical information describing the late phase of gravitational collapse.

Acknowledgements

We are grateful to Mihalis Dafermos and Ted Newman for fruitful discussions on these topics. We are also indebted to the two anonymous referees for important insights that have considerably improved the presentation and content of our work. We acknowledge financial support from CONICET, SeCyT-UNC, Foncyt and by the Agence Nationale de la Recherche; grant ANR-06-BLAN-0050. A.P. was supported by *l'Institut Universitaire de France*.

-
- [1] M. J. Reid. Is there a supermassive black hole at the center of the milky way? *Int.J.Mod.Phys.D*, 18:889, 2009, [arXiv:astro-ph/0808.2624].
 - [2] A. Mueller. Experimental evidence of black holes. *PoS P2GC*, page 017, 2006, [arXiv:astro-ph/0701228].
 - [3] A. E. Broderick, A. Loeb, and R. Narayan. The event horizon of Sagittarius A*. *Astrophys J.*, 701:1357, 2009, [arXiv:astro-ph/0903.1105].
 - [4] David Walter Pajerski and Ezra T. Newman. Trapped surfaces and the development of singularities. *J. Math. Phys.*, 12:1929 – 1937, 1971.
 - [5] A. Ashtekar and B. Krishnan. Isolated and dynamical horizons and their applications. *Living Rev. Rel.*, 7:10, (2004), <http://www.livingreviews.org/Irr-2004010>.
 - [6] Osvaldo M. Moreschi. Intrinsic angular momentum and center of mass in general relativity. *Class.Quantum Grav.*, 21:5409–5425, 2004.
 - [7] T. M. Adamo, C. N. Kozameh, and E. T. Newman. Null Geodesic Congruences, Asymptotically Flat Space-Times and Their Physical Interpretation. *Living Rev. Rel.*, 12:6, 2009.
 - [8] R. Geroch, A. Held, and R. Penrose. A space-time calculus based on pairs of null directions. *J. Math. Phys.*, 14:874–881, 1973.
 - [9] C. Gundlach, R. H. Price, and J. Pullin. Late time behavior of stellar collapse and explosions: 1. linearized perturbations. *Phys.Rev.D*, 49:883, 1994, [arXiv:gr-qc/9307009].